Joint Distribution

- We may be interested in probability statements of several RVs.
- Example: Two people A and B both flip coin twice. X: number of heads obtained by A. Y: number of heads obtained by B. Find $P(X > Y)$.
- Discrete case:
  Joint probability mass function: $p(x, y) = P(X = x, Y = y)$.
  - Two coins, one fair, the other two-headed. A randomly chooses one and B takes the other.

    $$X = \begin{cases} 1 & \text{A gets head} \\ 0 & \text{A gets tail} \end{cases} \qquad Y = \begin{cases} 1 & \text{B gets head} \\ 0 & \text{B gets tail} \end{cases}$$

    Find $P(X \geq Y)$.

- *Marginal probability mass function* of X can be obtained from the joint probability mass function, $p(x, y)$:

$$p_X(x) = \sum_{y:p(x,y)>0} p(x, y) \ .$$

Similarly:

$$p_Y(y) = \sum_{x:p(x,y)>0} p(x, y) \ .$$

- Continuous case:
  Joint probability density function $f(x,y)$:

$$P\{(X,Y) \in R\} = \int \int_R f(x,y)dxdy$$

- Marginal pdf:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

- Joint cumulative probability distribution function of X and Y

$$F(a,b) = P\{X \le a, Y \le b\} \quad -\infty < a, b < \infty$$

- Marginal cdf:

$$F_X(a) = F(a, \infty)$$

$$F_Y(b) = F(\infty, b)$$

- Expectation $E[g(X,Y)]$:

$= \sum_y \sum_x g(x,y)p(x,y)$ in the discrete case
$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy$ in the continuous case

- Based on joint distribution, we can derive

$$E[aX + bY] = aE[X] + bE[Y]$$

Extension:

$$\begin{aligned} & E[a_1 X_1 + a_2 X_2 + \cdots + a_n X_n] \\ = \; & a_1 E[X_1] + a_2 E[X_2] + \cdots + a_n E[X_n] \end{aligned}$$

- Example: $E[X]$, X is binomial with n, p:

$$X_i = \begin{cases} 1 & i\text{th flip is head} \\ 0 & i\text{th flip is tail} \end{cases}$$

$$X = \sum_{i=1}^{n} X_i \; , E[X] = \sum_{i=1}^{n} E[X_i] = np$$

- Assume there are n students in a class. What is the expected number of months in which at least one student was born. (Assume equal chance of being born in any month).

  Solution: Let X be the number of months some students are born. Let X_i be the indicator RV for the ith month in which some students are born. Then $X = \sum_{i=1}^{12} X_i$. Hence,

$$E(X) = 12E(X_1) = 12P(X_1 = 1) = 12 \cdot [1 - (\frac{11}{12})^n].$$

Independent Random Variables

- X and Y are *independent* if

$$P(X \leq a, Y \leq b) = P(X \leq a)P(Y \leq b)$$

- Equivalently: $F(a, b) = F_X(a)F_Y(b)$.
- Discrete: $p(x, y) = p_X(x)p_Y(y)$.
- Continuous: $f(x, y) = f_X(x)f_Y(y)$.
- Proposition 2.3: If X and Y are independent, then for function h and g, $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$.

Covariance

- Definition: *Covariance* of X and Y

$$Cov(X,Y) = E[(X - E(X))(Y - E(Y))]$$

- $Cov(X,X) = E[(X - E(X))^2] = Var(X)$.
- $Cov(X,Y) = E[XY] - E[X]E[Y]$.
- If X and Y are independent, $Cov(X,Y) = 0$.
- Properties:
  1. $Cov(X,X) = Var(X)$
  2. $Cov(X,Y) = Cov(Y,X)$
  3. $Cov(cX,Y) = cCov(X,Y)$
  4. $Cov(X,Y+Z) = Cov(X,Y) + Cov(X,Z)$

Sum of Random Variables

- If X_i's are independent, $i = 1, 2, ..., n$

$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$$

$$Var(\sum_{i=1}^{n} a_i X_i) = \sum_{i=1}^{n} a_i^2 Var(X_i)$$

- Example: Variance of Binomial RV, sum of independent Bernoulli RVs. $Var(X) = np(1-p)$.

Moment Generating Functions

- *Moment generating function* of a RV X is $\phi(t)$

$$\begin{aligned}\phi(t) &= E[e^{tX}] \\ &= \begin{cases} \sum_{x:p(x)>0} e^{tx}p(x) & X \text{ discrete} \\ \int_{-\infty}^{\infty} e^{tx}f(x)dx & X \text{ continuous} \end{cases}\end{aligned}$$

- Moment of X: the nth moment of X is $E[X^n]$.
- $E[X^n] = \phi^{(n)}(t) \mid t = 0$, where $\phi^{(n)}(t)$ is the nth order derivative.
- Example
  1. Bernoulli with parameter p: $\phi(t) = pe^t + (1-p)$, for any t.
  2. Poisson with parameter λ: $\phi(t) = e^{\lambda(e^t-1)}$, for any t.
- Property 1: Moment generation function of the sum of independent RVs:
  X_i, $i = 1, ..., n$ are independent, $Z = X_1 + X_2 + \cdots + X_n$,

$$\phi_Z(t) = \prod_{i=1}^{n} \phi_{X_i}(t)$$

- Property 2: Moment generating function uniquely determines the distribution.
- Example:
  1. Sum of independent Binomial RVs
  2. Sum of independent Poisson RVs
  3. Joint distribution of the sample mean and sample variance from a normal porpulation.

Important Inequalities

- Markov Inequality: If X is a RV that takes only non-negative values, then for any $a > 0$

$$P(X \geq a) \leq \frac{E[X]}{a} \ .$$

- Chebyshev's Inequality: If X is a RV with mean μ and variance σ^2, then for any value $k > 0$

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2} \ .$$

- Examples: obtaining bounds on probabilities.

Strong Law of Large Numbers

- Theorem 2.1 (Strong Law of Large Numbers): Let X_1, X_2, ..., be a sequence of independent random variables having a common distribution. Let $E[X_i] = \mu$. Then, with probability 1

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \to \mu \text{ as } n \to \infty$$

Central Limit Theorem

- Theorem 2.2 (Central Limit Theorem): Let X_1, X_2, ..., be a sequence of independent random variables having a common distribution. Let $E[X_i] = \mu$, $Var[X_i] = \sigma^2$. Then the distribution of

$$\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal as $n \to \infty$. That is

$$P\{\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq z\}$$

$$\to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx = \Phi(z)$$

- Example: estimate probability.

  1. Let X be the number of times that a fair coin flipped 40 times lands heads. Find $P(X = 20)$.
  2. Suppose that orders at a restaurant are iid random variables with mean $\mu = 8$ dollars and standard deviation $\sigma = 2$ dollars. Estimate the probability that the first 100 customers spend a total of more than \$840. Estimate the probability that the first 100 customers spend a total of between \$780 and \$820.